PROJECTIVE LINKING AND BOUNDARIES OF POSITIVE HOLOMORPHIC CHAINS IN PROJECTIVE MANIFOLDS, PART II

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Abstract

Part I introduced the notion of the *projective linking number* $\mathrm{Link}_{\mathbf{P}}(\Gamma, Z)$ of a compact oriented real submanifold Γ of dimension $2p-1$ in complex projective n-space $\mathbf{P}^n$ with an algebraic subvariety $Z \subset \mathbf{P}^n - \Gamma$ of codimension p. It is shown here that a basic conjecture concerning the projective hull of real curves in $\mathbf{P}^2$ implies the following result:

Γ *is the boundary of a positive holomorphic p-chain of mass $\leq \Lambda$ in $\mathbf{P}^n$ if and only if the* $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \geq -\Lambda$ *for all algebraic subvarieties Z of codimension-p in $\mathbf{P}^n - \Gamma$.*

where $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) = \mathrm{Link}_{\mathbf{P}}(\Gamma, Z)/p! \deg(Z)$. An analogous result is implied in any projective manifold X.

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*Partially supported by the N.S.F.

1. Introduction

In Part I we introduced a linking pairing for certain cycles in projective space as follows. Suppose that $\Gamma \subset \mathbf{P}^n$ is a compact oriented submanifold of dimension $2p-1$, and let $Z \subset \mathbf{P}^n - \Gamma$ be an algebraic subvariety of codimension p. The **projective linking number** of Γ with Z is defined to be

$$\mathrm{Link}_{\mathbf{P}}(\Gamma, Z) \;\equiv\; N \bullet Z - \deg(Z) \int_N \omega^p$$

where ω is the standard Kähler form on $\mathbf{P}^n$ and N is any integral 2p-chain with $\partial N = \Gamma$ in $\mathbf{P}^n$. This definition is independent of the choice of N. The associated **reduced linking number** is defined to be

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \;\equiv\; \frac{1}{p!\deg(Z)} \mathrm{Link}_{\mathbf{P}}(\Gamma, Z)$$

The basic result proved here is the following.

Theorem 1.1. *Let $\Gamma \subset \mathbf{P}^n$ be a compact oriented real analytic submanifold of dimension $2p-1$ with possible integer multiplicities on each component. If Conjecture B holds (See below), then the following are equivalent:*

(1) Γ is the boundary of a positive holomorphic p-chain of mass $\leq \Lambda$ in $\mathbf{P}^n$.

(2) $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \;\geq\; -\Lambda$ for all algebraic subvarieties Z of codimension p in $\mathbf{P}^n - \Gamma$.

A compact subset $K \subset \mathbf{P}^n$ is called **stable** if the best constant function is bounded on the projective hull $\widehat{K}$ (See [$\mathrm{HL}_{3,4}$]). It is known that for any stable real analytic curve $\gamma \subset \mathbf{P}^n$, the set $\widehat{\gamma} - \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbf{P}^n - \gamma$, [HLW]. Conjecture A from Part I is the statement that any compact real analytic curve $\gamma \subset \mathbf{P}^n$ is stable. Even more likely is the following.

Conjecture B. Let $\gamma \subset \mathbf{P}^2$ be a compact embedded real analytic curve such that for some choice of orientation and positive integer multiplicity on each component, condition (2) above is satisfied. Then γ is stable.

In Part I the conclusion of Theorem 1.1 was established for any stable real analytic curve $\Gamma \subset \mathbf{P}^n$ (with orientation and multiplicity on each component). The main point of Part II is to prove that this result for $p = 1$ implies the result for all $p > 1$, provided one can drop the stability hypothesis in the $p = 1$ case.

Note incidentally that there is no assumption of maximal complexity on the cycle Γ.

Theorem 1.1 represents a projective analogue of a result of H. Alexander and J. Wermer [AW].

If the cycle Γ in Theorem 1.1 bounds a holomorphic p-chain T, then there is a unique such chain T_0 of least mass with $dT_0 = \Gamma$. (All others are obtained by adding positive algebraic p-cycles to T_0.)

Corollary 1.2. *Let Γ be as in Theorem 1.1 and suppose that $\Gamma = dT$ for some positive holomorphic p-chain T. Then T is the chain of least mass with boundary Γ if and only if*

$$\inf_Z \left\{ \frac{T \bullet Z}{\deg Z} \right\} \ = \ 0$$

where the infimum is taken over all positive algebraic $(n-p)$-cycles in $\mathbf{P}^n - \Gamma$.

The linking hypothesis (2) in Theorem 1.1 can be replaced by other hypotheses. This is discussed in §3. Another interesting consequence of Theorem 1.1 is the following result, whose proof follows exactly the lines given in Part I for the case $p = 1$.

Theorem 1.3. *Let $M \subset \mathbf{P}^n$ be a compact embedded real analytic submanifold of dimension $2p-1$ and assume Conjecture B. Then a class $\tau \in H_{2p}(\mathbf{P}^n, M; \mathbf{Z})$ contains a positive holomorphic chain T (with $\operatorname{supp} dT \subseteq M$) if and only if*

$$\tau \bullet u \ \geq \ 0$$

for all classes $u \in H_{2n-2p}(\mathbf{P}^n - M; \mathbf{Z})$ which are represented by positive algebraic cycles.

Theorem 1.1 and many of its consequences carry over to general projective manifolds. This is done in §4.

We recall our convention that $d^C \ = \ \frac{i}{2\pi}(\overline{\partial} - \partial)$

2. The Projective Alexander-Wermer Theorem.

Let Γ be a compact smooth oriented submanifold of dimension $2p-1$ in $\mathbf{P}^n$. We recall that (even if Γ is only class C^1) any irreducible complex analytic subvariety $V \subset \mathbf{P}^n - \Gamma$ has finite Hausdorff $2p$-measure and defines a current $[V]$ of dimension $2p$ in $\mathbf{P}^n$ by integration on the canonically oriented manifold of regular points. Furthermore, the boundary of this current is of the form $d[V] = \sum_j \epsilon_j \Gamma_j$ where $\Gamma_1, ..., \Gamma_\ell$ represent the connected components of Γ and $\epsilon_j = 1, 0,$ or -1. (See [H] for example.) We now allow Γ to carry positive integer multiplicities on each component, so it is of the form $\Gamma = \sum_j m_j \Gamma_j$.

Definition 2.1. By a **positive holomorphic p-chain with boundary** Γ we mean a finite sum $T = \sum_k n_k [V_k]$ where each $n_k \in \mathbf{Z}^+$ and each $V_k \subset \mathbf{P}^n - \Gamma$ is an irreducible subvariety of dimension p, so that

$$dT \ = \ \Gamma \qquad \text{(as currents on } \mathbf{P}^n)$$

By the **mass** of such a chain $T = \sum_k n_k [V_k]$ we mean its weighted volume: $\mathbf{M}(T) \equiv \sum_k n_k \mathcal{H}^{2p}(V_k) = T(\Omega_p)$ where $\mathcal{H}^{2p}$ denotes Hausdorff $2p$-measure and

$$\Omega_p \ \equiv \ \frac{1}{p!}\omega^p$$

Proposition 2.2. *Suppose T is a positive holomorphic p-chain with boundary Γ as above. Then*

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \;\geq\; -\mathbf{M}(T)$$

for all positive algebraic cycles Z with support in $\mathbf{P}^n - \Gamma$.

Proof. Note that since $dT = \Gamma$ we have

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) = \frac{T \bullet Z}{p!\,\mathrm{deg} Z} - T\left(\Omega_p\right) \geq -T\left(\Omega_p\right) = -\mathbf{M}(T)$$

since $T \bullet Z \geq 0$ by the positivity of T and Z. ■

Note that Proposition 2.2 holds for positive holomorphic chains with quite general boundaries Γ. This brings us to the main result.

Theorem 2.3. *Under the assumption of Conjecture B the following are equivalent.*

(1) $\Gamma = dT$ *where T is a positive holomorphic p-chain in $\mathbf{P}^n$ with $\mathbf{M}(T) \leq \Lambda$*

(2) $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \;\geq\; -\Lambda$ *for all $(n-p)$-dimensional algebraic varieties $Z \subset \mathbf{P}^n - \Gamma$.*

Proof. Proposition 2.2 states that (1) $\Rightarrow$ (2). For the converse we shall show that the linking condition persists for hyperplane slices, and then proceed by induction on dimension.

Proposition 2.4. *Suppose that Γ satisfies the Λ-linking condition (2) in $\mathbf{P}^n$. If $H \cong \mathbf{P}^{n-1}$ is a hyperplane which intersects Γ transversely, then $\Gamma_H \equiv \Gamma \cap H$ satisfies the Λ'-linking condition in H where*

$$\Lambda' \;=\; p\Lambda + \int_{\Gamma} d^C u \wedge \Omega_{p-1}$$

and $u = \log(|Z_0|/\|Z\|)$ where Z_0 is the linear function defining H.

Proof. Since bordism and homology agree in $\mathbf{P}^n$ there exists a compact oriented $2p$-manifold N with boundary and a smooth map $f : N \to \mathbf{P}^n$ such that f is an immersion near ∂N and

$$f\big|_{\partial N} \colon \partial N \to \Gamma \qquad \text{is an oriented diffeomorphism.}$$

Since Γ is transversal to H, f is also transversal to H near the boundary. By standard transversality theory we can perturb f, keeping it fixed near the boundary, so that it is everywhere transversal to H. Let $N_H \equiv f^{-1}(H)$ oriented by N and the normal bundle to H, and let $f_H : N_H \to H$ be the restriction of f. Then $(f_H)_*[N_H]$ defines a $(2p-2)$-dimensional current in H with boundary Γ_H. We denote this current simply by $[N_H]$.

Suppose now that $Z \subset H - \Gamma_H$ is an $(n-p)$-dimensional algebraic subvariety. We may assume, again by a small perturbation, that f misses the singular set of Z and is transversal to $\mathrm{Reg}(Z)$. It is then straightforward to check that

$$[N] \bullet Z \;=\; [N_H] \bullet_H Z \tag{2.1}$$

where "$\bullet_H$" denotes the intersection pairing in H (defined as in §3 of Part I).

By assumption we have that

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \;=\; \frac{1}{p!}\left\{\frac{1}{\deg Z}([N]\bullet Z) - \int_N \omega^p\right\} \;\geq\; -\Lambda. \tag{2.2}$$

Now the function u above satisfies the Poincaré-Lelong equation

$$dd^C u \;=\; H - \omega. \tag{2.3}$$

Substituting (2.1) and (2.3) into (2.2) gives

$$\begin{aligned}
\frac{1}{\deg Z}([N_H]\bullet_H Z) - \int_N (H - dd^C u)\wedge\omega^{p-1} \;=& \\
&= \frac{1}{\deg Z}([N_H]\bullet_H Z) - \int_{N_H}\omega^{p-1} + \int_N dd^C u\wedge\omega^{p-1} \\
&= \widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma_H, Z)(p-1)! + \int_\Gamma d^C u\wedge\omega^{p-1} \\
&\geq -\Lambda p!
\end{aligned}$$

where the first equality is straightforwardly justified using transversality. ■

COROLLARY 2.5. *Assume that (2) ⇒ (1) for all manifolds Γ of dimension $2p-3$ in projective space. Suppose that Γ is a $(2p-1)$-manifold satisfying (2) and that H is a hyperplane transversal to Γ. Then there exists $\Lambda' > 0$ so that $\Gamma_{H'} \equiv \Gamma\cap H'$ bounds a positive holomorphic $(p-1)$-chain of mass $\leq \Lambda'$ for all hyperplanes H' in a neighborhood U of H.*

Proof. If H is transversal to Γ then so are all hyperplanes H' in a neighborhood of H. Furthermore, the integral $\int_\Gamma d^C(u_{H'})\wedge\omega^{p-1}$ depends continuously on H' in that neighborhood, where $u_{H'} = \log(|(A_{H'}, Z)|/\|Z\|)$ and $A_{H'}$ is a continuous choice of vectors with $H' = \{[Z]\in\mathbf{P}^n : (A_{H'}, Z) = 0\}$. It follows that the constant Λ' in Proposition 2.4 is uniformly bounded below in a neighbohhood of H. One then applies the inductive hypothesis. ■

PROPOSITION 2.6. *Let U be the neighborhood given in Corollary 2.5. For each hyperplane H' in U let $T_{H'}$ be the positive holomorphic $p-1$ chain of least mass with $dT_{H'} = \Gamma_{H'}$. Then $T_{H'}$ is uniquely determined by H' and the mapping $H'\mapsto T_{H'}$ is continuous on U.*

Proof. We first prove uniqueness. For future reference we formulate this result for Γ instead of $\Gamma_{H'}$.

LEMMA 2.7. *Suppose Γ bounds a positive holomorphic p-chain. Then the positive holomorphic p-chain of least mass with boundary Γ is unique.*

Proof. Suppose that $T = \sum_i n_i[V_i]$ and $T' = \sum_j n'_j[V'_j]$ are positive holomorphic p-chains of least mass having the same boundary $dT = dT' = \Gamma$ in $\mathbf{P}^n$. By the least mass hypothesis we know that

$$d[V_i] \;\neq\; 0 \qquad \text{and} \qquad d[V'_j] \;\neq\; 0 \qquad \text{for all } \; i, j. \tag{2.4}$$

In fact $d[V_i]$ and $d[V_j']$ each consist of a finite number of oriented connected components of Γ, each with multiplicity one. Let Γ_1 be an oriented connected component of Γ. (If Γ has multiplicity greater than 1 along Γ_1, we ignore that multiplicity for the moment.) Then there must exists a component of T, say V_1, such that Γ_1 forms part of the oriented boundary dV_1. Similarly there is a component, say V_1' of T' such that Γ_1 is part of dV_1'. By boundary regularity [HL$_1$], and local and global uniqueness these two irreducible subvarieties of $\mathbf{P}^n - \Gamma$ must coincide. Hence, $S \equiv T - V_1$ and $S' \equiv T' - V_1'$ are positive holomorphic p chains with $dS = dS'$. By continuing this process one of the two chains will eventually be reduced to zero. However, the other must also be 0 since its boundary is zero and its remaining components satisfy condition (2.4) ■

To prove continuity it will suffice to show that every convergent sequence $H_j \to H$ has a subsequence such that $T_{H_j} \to T_H$. By the local uniform bound on the mass, the fact that $dT_{H_j} \to dT_H = \Gamma_H$, and the compactness of positive holomorphic chains, we know that there is a subsequence which converges to some positive holomorphic chain T with boundary Γ_H. We then apply the uniquness. ■

We recall that Γ is said to be **maximally complex** if

$$\dim_{\mathbf{C}}(T_x\Gamma \cap JT_x\Gamma) \ = \ p-1 \qquad \text{for all} \ \ x \in \Gamma$$

where J is the almost complex structure on $\mathbf{P}^n$.

PROPOSITION 2.8. *Assume Conjecture B. If Γ satisfies (2), then Γ is maximally complex.*

Proof. The result is trivial when $\dim\gamma = 1$ so we first consider the case $\Gamma = \Gamma^3 \subset \mathbf{P}^3$. We want to show that $\int_\Gamma \alpha = 0$ for all $(3,0)$-forms α on $\mathbf{P}^3$.

Choose a line $L \cong \mathbf{P}^1 \subset \mathbf{P}^3$ with $\Gamma \cap L = \emptyset$ and a linear projection $\pi : \mathbf{P}^3 - L \to \mathbf{P}^1$. Fix a point $x_\infty \in \mathbf{P}^1$ and choose affine coordinates (z_0, z_1, z_2) on $\pi^{-1}(\mathbf{P}^1 - \{x_\infty\}) \cong \mathbf{C}^3$. We shall show that

$$\int_\Gamma g(z_0) dz_0 \wedge dz_1 \wedge dz_2 = 0 \tag{2.5}$$

for all $g \in C_0^\infty(\mathbf{C})$. Such forms, taken over all possible choices above, are dense in $\mathcal{E}^{3,0}$ on a neighborhood of Γ. Hence, $\int_\Gamma \alpha = 0$ for all $(3,0)$-forms α, which implies that Γ is maximally complex.

To prove (2.5) we choose a 4-chain N with compact support in $\mathbf{P}^3 - L$ such that $dN = \Gamma$. Then (2.5) can be rewritten as

$$\int_N \frac{\partial g}{\partial \overline{z}_0} \, dz_0 \wedge d\overline{z}_0 \wedge dz_1 \wedge dz_2 \ = \ 0$$

for all $g \in C_0^\infty(\mathbf{C})$. For this it will suffice to prove that

$$(N \wedge dz_1 \wedge dz_2, \pi^*\eta) \ = \ (\pi_*(N \wedge dz_1 \wedge dz_2), \eta) \ = \ 0$$

for any (1,1)-form η with compact support in $\mathbf{C}$. For this it will suffice to consider $\eta = \delta(z_0 - t) dz_0 \wedge d\overline{z}_0$ for $t \in \mathbf{C}$, in other words we want to show that the slice at t:

$$\{\pi_*(N \wedge dz_1 \wedge dz_2)\}_t \ = \ \pi_*\{N_t \wedge dz_1 \wedge dz_2\} \ = \ \int_{N_t} dz_1 \wedge dz_2 \ = \ 0 \tag{2.6}$$

for all $t \in \mathbf{C}$.

Observe now that $dN_t = \Gamma_t$, the slice of Γ by π at t, and by Proposition 2.4 this Γ_t satisfies the projective linking condition (2). Hence by Theorem 6.1 in [HL$_4$] and our hypothesis that Γ_t is stable, we conclude that $\Gamma_t = dT_t$ where T_t is a positive holomorphic 1-chain in $\mathbf{P}^2 =$ the closure of $\pi^{-1}(t)$. Thus our desired condition (2.6) is established by the following.

Lemma 2.9. *Let γ be a curve, or in fact any rectifiable 1-cycle with compact support in $\mathbf{C}^2 \subset \mathbf{P}^2$. Suppose $\gamma = dT$ where T is a positive holomorphic chain in $\mathbf{P}^2$. Then for any $S \in \mathcal{D}'_{2,\mathrm{cpt}}(\mathbf{C}^2)$ with $dS = \gamma$, one has $S(dz_1 \wedge dz_2) = 0$.*

Proof. It suffices to construct one current S with these properties. We can assume that the line at infinity $\mathbf{P}^1_\infty = \mathbf{P}^2 - \mathbf{C}^2$ meets $\operatorname{supp} T$ only at regular points and is transversal there. (The general result follows directly.) Choose $x \in \operatorname{supp} T \cap \mathbf{P}^1_\infty$ and let L be the tangent line to supp T at x. Then after an affine transformation of the (z_1, z_2)-coordinates, we may assume $L \cong z_1$-axis. This transformation can be chosen with determinant one, so the form $dz_1 \wedge dz_2$ remains unchanged.

Near the point x, the current T is given by a positive multiple of the graph $\Sigma_R \equiv \{(z_1, f(\frac{1}{z_1})) : |z_1| \geq R\}$ where f is holomorphic in the disk of radius $1/R$ and satisfies

$$f(0) \;=\; f'(0) \;=\; 0.$$

In particular we have that

$$\lim_{z_1 \to 0} z_1 f(1/z_1) \;=\; 0. \tag{2.7}$$

We now modify T by replacing (the appropriate multiple of) Σ_R with the current $L_R + U_R$ where

$$L_R \;\equiv\; \{(z_1, 0) : |z_1| \leq R\} \qquad \text{and} \qquad U_R \;\equiv\; \{(z_1, tf(1/z_1)) : |z_1| = R \;\text{ and }\; 0 \leq t \leq 1\}$$

with orientations chosen so that $d(L_R + U_R) = d\Sigma_R$. Observe that

$$\int_{L_R+U_R} dz_1 \wedge dz_2 \;=\; \int_{U_R} dz_1 \wedge dz_2 \;=\; -\int_{dU_R} z_2 dz_1 \;=\; \int_0^{2\pi} f(e^{i\theta}/R)\frac{ie^{i\theta}}{R}d\theta \;\to\; 0$$

as $R \to \infty$ by (2.7).

Carrying out this procedure at each point of $\operatorname{supp} T \cap \mathbf{P}^1_\infty$ we obtain a current $T(R_1, ..., R_\ell)$ with compact support in $\mathbf{C}^2$ and with $dT(R_1, ..., R_\ell) = \gamma$. Since $dz_1 \wedge dz_2$ is closed we have

$$S(dz_1 \wedge dz_2) \;=\; T(R_1, ..., R_\ell)(dz_1 \wedge dz_2) \;=\; \sum_{k=1}^{\ell} \int_{U_{R_k}} dz_1 \wedge dz_2 \;\to\; 0$$

as $R_1, ..., R_\ell \to \infty$. ■

We have now proved the proposition for 3-folds in $\mathbf{P}^3$. The result for 3-folds in $\mathbf{P}^n$ follows by considering the family of projections $\mathbf{P}^n - - - > \mathbf{P}^3$ which are well defined on Γ.

The result for general Γ follows by intersecting with hyperplanes and applying Proposition 2.4. ■

We now show that Γ bounds a positive holomorphic chain by applying the main result in [DH]. Let $L \subset \mathbf{P}^n$ be a linear subspace of (complex) codimension $p-1$ which is transversal to Γ. We assume that L meets every component of Γ. This can be arranged by taking a Veronese embedding $\mathbf{P}^n \subset \mathbf{P}^N$ of sufficiently high degree, and working with linear subspaces there. One checks that the projective linking numbers of Γ are also bounded in $\mathbf{P}^N$. By applying Proposition 2.4 inductively we see that for all linear subspaces L' in a neighborhood of L, the intersections $\Gamma_{L'} = \Gamma \bullet L'$ satisfy the projective linking condition for oriented curves with multiplicities. Therefore by [HL_4] and our assumption of Conjecture B, each slice $\Gamma_{L'}$ bounds a positive holomorphic 1-chain. With this property and maximal complexity, It follows directly from [DH] that Γ bounds a holomorphic p-chain. The unique minimal such chain T will be supported in the subvariety $\mathbf{P}^n \subset \mathbf{P}^N$ because Γ is. Furthermore, T must be positive. If not, there would be a negative component, say T_0. Since T is minimal, $dT_0 \neq 0$. Now L must meet T_0 since it meets all components of Γ. It follows that the minimal holomorphic 1-chain with boundary Γ_L is not positive – a contradiction.

Note .This last paragraph could be replaced by an argument based on the results in [HL_2].

So we have proved that $\Gamma = dT$ where T is a positive holomorphic p-chain. We may assume that T is the unique such chain of least mass. It remains to prove that $\mathbf{M}(T) \leq \Lambda$.

Suppose not. Then

$$\mathbf{M}(T) \ = \ T(\Omega_p) \ = \ \Lambda + r \tag{2.8}$$

for $r > 0$, and we see that

$$\begin{aligned} \frac{T \bullet Z}{p! \deg Z} \ &= \ \frac{T \bullet Z}{p! \deg Z} - T(\Omega_p) + T(\Omega_p) \\ &= \ \widetilde{\text{Link}}_{\mathbf{P}}(\Gamma, Z) + T(\Omega_p) \ \geq \ -\Lambda + \Lambda + r = r \end{aligned}$$

for all algebraic subvarieties Z of codimension p in $\mathbf{P}^n - \Gamma$. Hence, it will suffice to prove that

$$\inf_Z \left\{ \frac{T \bullet Z}{\deg Z} \right\} \ = \ 0 \tag{2.9}$$

where Z varies over the codimension p subvarieties of $\mathbf{P}^n - \Gamma$.

Lemma 2.10. *Let T be a positive holomorphic p-chain in $\mathbf{P}^n$ with boundary Γ. Then T is the unique such chain of least mass if and only if every irreducible component of* $\operatorname{supp} T$ *in $\mathbf{P}^n - \Gamma$ has a non-empty boundary (consisting of components of Γ).*

Proof. If T has least mass, it can have no components with boundary zero, since one could remove these components and thereby reduce the mass without changing the boundary. If T is not of least mass, let T_0 be the least mass solution and note that $d(T - T_0) = 0$. It follows from [HS] (actually, elementary arguments involving local uniqueness at the boundary will suffice here) that $T - T_0 = S - S_0$ where S and S_0 are positive algebraic p-cycles,

i.e., $dS = dS_0 = 0$. We assume that S and S_0 have no components in common. From the equation $T + S_0 = T_0 + S$ and the uniqueness of the decomposition of analytic subvarieties into irreducible components, we see that the components of S must be components of T and similarly the components of S_0 must be components of T_0. Since T_0 is least mass, we have $S_0 = 0$ and $T = T_0 + S$. Since T is not the least mass solution, $S \neq 0$. ■

Our proof now proceeds by induction on p. We assume that condition (2.9) holds for least mass chains of dimension $< p$, and we shall show that it hold in dimension p. The case $p = 1$ has already been established in Part I, Corollary 6.8.

We return to our positive holomorphic p-chain T of least mass. A positive divisor D in $\mathbf{P}^n$ is defined to be **totally transverse** to T if:

(i) D is smooth,

(ii) D meets every component of $dT = \Gamma$, and each of these intersections is transverse,

(iii) D is transversal to every stratum of the singular stratification of $\operatorname{supp} T$.

This condition is open, and it is non-empty for divisors of sufficiently high degree. Let D be such a divisor with degree d. Let $\mathbf{P}^n \subset \mathbf{P}^N$ be the order d Veronese embedding, and let $H_0 \subset \mathbf{P}^N$ be the hyperplane with $D = H_0 \bullet \mathbf{P}^n$. Then H_0 is totally transverse to T in $\mathbf{P}^N$. Assume there exists H in a neighborhood of H_0 which is totally transverse to T and such that $T_H \equiv H \bullet T$ has no irreducible components with boundary zero. Then there exists a sequence $\{Z_j\}_{j \geq 0}$ of subvarieties of codimension $p - 1$ in $H - \Gamma$ such that

$$\lim_{j \to \infty} \left\{ \frac{T_H \bullet_H Z_j}{\deg Z_j} \right\} \;=\; 0.$$

Note that $T_H \bullet_H Z_j = T \bullet_{\mathbf{P}^N} Z_j$, and so

$$\lim_{j \to \infty} \left\{ \frac{T \bullet_{\mathbf{P}^N} Z_j}{\deg Z_j} \right\} \;=\; 0.$$

Now by a small perturbation we may assume that each Z_j is transversal to $\mathbf{P}^n$. Then $W_j \equiv Z_j \cap \mathbf{P}^n$ is a subvariety of codimension p in $\mathbf{P}^n$ with $T \bullet_{\mathbf{P}^n} W_j = T \bullet_{\mathbf{P}^N} Z_j$ and with degree $\deg W_j = d^p \deg Z_j$. It follows that

$$\lim_{j \to \infty} \left\{ \frac{T \bullet_{\mathbf{P}^n} W_j}{\deg W_j} \right\} \;=\; 0$$

as desired.

So we are done unless every totally transverse hyperplane section T_H has components with zero boundary. We will show that this cannot happen for H in a neighborhood of H_0. For the sake of clarity, we assume first that each component of $\operatorname{supp} T$ is smooth. Then by transversality each component of $\operatorname{supp} T_H$ is smooth. Suppose there exists a component V of $\operatorname{supp} T_H$ which is without boundary. Then V will be contained in one of the components of $\operatorname{supp} T$, which we denote by W. Note that W is a smooth submanifolds with real analytic boundary.

Now let V_ϵ be a neighborhood of V in W. Since V_ϵ is a subvariety defined in a neighborhood of a hyperplane, it extends to an irreducible **algebraic** subvariety Y of dimension p. It follows that W is a subdomain with real analytic boundary in Y.

Now the generic hyperplane section of an irreducible variety is again irreducible when $p > 1$ (See [Ha, Prop. 18.10]). It follows that $W \cap H = V$. However, this is impossible since H meets every component of Γ and so it must meet the components of Γ which are contained in W.

When $\operatorname{supp} T$ is not smooth the argument is similar. By total transversality and our assumption, there exists a component V of $\operatorname{supp} T_H$ (for generic H) with no boundary and with the property that V extends to an irreducible p-dimensional subvariety V_ϵ in a neighborhood of H. By [HL_1, Thm. 9.2], V_ϵ extends to an irreducible algebraic subvariety Y of dimension p in $\mathbf{P}^N$. The remainder of the argument is the same. ∎

There is an "affine" version of Theorem 2.3 parallel to the "affine" version (Theorem 6.6) of Theorem 6.1 in Part I. However the reader should note that the hypothesis that Γ is contained in some affine chart is not satisfied generically when $\dim(\Gamma) > 1$.

Theorem 2.10. *Let Γ is as in Theorem 2.3 and suppose $\Gamma \subset \mathbf{C}^n$ for some affine chart $\mathbf{C}^n \subset \mathbf{P}^n$. Assuming Conjecture B, the following are equivalent:*

(1) There exists a constant Λ so that the classical linking number satisfies

$$\text{Link}_{\mathbf{C}^n}(\Gamma, Z) \;\geq\; -\Lambda\, p!\, \deg Z$$

for all $(n-p)$-dimensional algebraic varieties $Z \subset \mathbf{C}^n - \Gamma$.

(2) $\Gamma = dT$ where T is a positive holomorphic p-chain in $\mathbf{P}^n$ with

$$\mathbf{M}(T) \leq \Lambda + \frac{1}{p}\int_\Gamma d^C \log\sqrt{1+\|z\|^2} \wedge \Omega_{p-1}.$$

Proof. We recall that in terms of the affine coordinate z on $\mathbf{C}^n$, the Kaehler form is given by $\omega = \frac{1}{2} dd^c \log(1+\|z\|^2)$. Now let N be a $2p$-chain in $\mathbf{C}^n$ with $dN = \Gamma$. Then $\widetilde{\text{Link}}_{\mathbf{P}}(\Gamma, Z)p! = \frac{1}{\deg Z} N \bullet Z - \int_N \omega^p = \frac{1}{\deg Z}\text{Link}_{\mathbf{C}^n}(\Gamma, Z) - \int_\Gamma d^C \log\sqrt{1+\|z\|^2} \wedge \omega^{p-1}$. The result now follows directly from Theorem 2.3. ∎

3. Theorems for General Projective Manifolds.

The results established above generalize from $\mathbf{P}^n$ to any projective manifold. Let X be a compact complex n-manifold with a positive holomorphic line bundle λ. Fix a hermitian metric on λ with curvature form $\omega > 0$, and give X the Kähler metric associated to ω. Let Γ be a $(2p-1)$-cycle on X with properties as in §2 (i.e., an oriented $(2p-1)$-dimensional submanifold with integral weights), and assume $[\Gamma] = 0$ in $H_{2p-1}(X; \mathbf{Z})$.

Definition 3.1. Let Z be a positive algebraic $(n-p)$-cycle on X which has cohomology class $\ell[\omega^p]$ for some $\ell \geq 1$. If Z does not meet Γ, we can define the **linking number** and the **reduced linking number** by

$$\text{Link}_\lambda(\Gamma, Z) \;\equiv\; N \bullet Z - \ell \int_N \omega^p \qquad \text{and} \qquad \widetilde{\text{Link}}_\lambda(\Gamma, Z) \;\equiv\; \frac{1}{\ell\, p!}\,\text{Link}(\Gamma, Z)$$

respectively, where N is any $2p$-chain in X with $dN = \Gamma$ and where the intersection pairing $\bullet$ is defined as in §3 of Part I with $\mathbf{P}^n$ replaced by X.

To see that this is well-defined suppose that N' is another $2p$-chain with $dN' = \Gamma$. Then $(N-N')\bullet Z - \ell\int_{N-N'}\omega^p = (N-N')\bullet(Z-\ell[\omega^p]) = 0$ because $Z-\ell\omega^p$ is cohomologous to zero in X.

THEOREM 3.2. *Under the assumption of Conjecture B the following are equivalent:*

(1) $\Gamma = dT$ *where T is a positive holomorphic p-chain on X with mass $\leq \Lambda$.*

(2) $\widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) \geq -\Lambda$ *for all positive algebraic $(n-p)$-cycles $Z \subset X - \Gamma$ of cohomology class $\ell[\omega]^p$ for $\ell \in \mathbf{Z}^+$.*

Proof. That (1) $\Rightarrow$ (2) follows as in the proof of Proposition 2.2. In fact this shows that (2) holds with no restriction on the cohomology class of Z.

For the converse we may assume (by replacing λ with λ^m if necessary) that the full space of sections $H^0(X, \mathcal{O}(\lambda))$ gives an embedding $X \subset \mathbf{P}^N$. An algebraic subvariety $\widetilde{Z} \subset \mathbf{P}^N$ of codimension p is said to be *transversal* to X if each level of the singular stratification of $\widetilde{Z}$ is transversal to X. More generally, a positive algebraic cycle $\widetilde{T} = \sum_\alpha n_\alpha \widetilde{Z}_\alpha$ of codimension p is *transversal* to X if each Z_α is. Such cycles are dense in the Chow variety of all positive algebraic $(N-p)$-cycles in $\mathbf{P}^N$. This follows from the Transversality Theorem for Families applied to the family $\mathrm{GL}_{\mathbf{C}}(N+1)\cdot\widetilde{Z}$ and the submanifold X in $\mathbf{P}^N$ (cf. [HL$_1$, App. A]).

It is straightforward to check that if $\widetilde{Z}$ is transversal to X, then $Z = \widetilde{Z} \cap X$ is an algebraic subvariety of codimension p in X with cohomology class $\ell[\omega^p]$ where $\ell = \deg \widetilde{Z}$. Let $\widetilde{Z}$ be such a cycle with the property that $\widetilde{Z} \cap \Gamma = \emptyset$. Let N be a 2p-chain in X with boundary Γ which meets $Z = \widetilde{Z} \cap X$ transversely at regular points. Then local computation of intersection numbers shows that $N \bullet_X Z = N \bullet_{\mathbf{P}^N} \widetilde{Z}$. Consequently, hypothesis (2) implies that

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, \widetilde{Z}) \;=\; \widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) \geq -\Lambda.$$

Since this holds for a dense set of subvarieties of $\mathbf{P}^n - \Gamma$ it holds for all such subvarieties. Theorem 2.3 now implies that Γ bounds a holomorphic chain T in $\mathbf{P}^N$. Since Γ is supported in X, so also is T. ∎

4. Variants of the Main Hypothesis

The linking hypothesis (2) in Theorem 3.2 can be replaced by several quite different conditions thereby yielding several geometrically distinct results. In this section we shall examine these conditions.

Let X, λ, ω and Γ all be as in §3. Suppose $Z \subset X$ is an algebraic subvariety of codimension-p with cohomology class $\ell[\omega^p]$. Then a **spark** associated to Z with curvature $\ell\omega^p$ is a current $\alpha \in \mathcal{D}'^{2p-1}(X)$ which satisfies the *spark equation*

$$d\alpha \;=\; Z \;-\; \ell\omega^p. \tag{4.1}$$

Such sparks form the basis of (one formulation of) the theory of differential characters (cf. [HLZ]). Two sparks α, α' satisfying (4.1) will be called **commensurate** if $\alpha' = \alpha + d\beta$ for $\beta \in \mathcal{D}'^{2p-2}(X)$.

A. λ-Winding Numbers. Suppose now that $\Gamma \cap Z = \emptyset$. Up to commensurability we may assume α is smooth in a neighborhood of Γ (See [HLZ, Prop. 4.2]). The **reduced λ-winding number** can then be defined as

$$\widetilde{\mathrm{Wind}}_\lambda(\Gamma, \alpha) \;\equiv\; \frac{1}{\ell\, p!} \int_\Gamma \alpha.$$

It follows from the spark equation (4.1) that

$$\widetilde{\mathrm{Wind}}_\lambda(\Gamma, \alpha) \;=\; \widetilde{\mathrm{Link}}_\lambda(\Gamma, Z). \tag{4.2}$$

In particular, this winding number is independent of the choice of α. This can be directly verified using deRham theory and the fact that $[\Gamma] = 0$ in $H_{2p-1}(X; \mathbf{Z})$.

B. Positivity. Recall that a smooth (p,p)-form ψ on X is called **weakly positive** if $\psi(\xi) \ \geq\ 0$ for all simple $2p$-vectors ξ representing canonically oriented complex tangent p-planes to X. A current $T \in \mathcal{D}'_{p,p}(X)$ of bidimension p,p is called **positive** if $T(\psi) \geq 0$ for all weakly positive (p,p)-forms ψ on X. In this case we write $T \geq 0$.

C. Algebraic Homology. Set $H^+_{2k} \equiv \{z \in H_{2k}(X - |\Gamma|; \mathbf{Z}) : (\omega^k, z) \geq 0\}$, where $|\Gamma| \equiv \operatorname{supp} \Gamma$, and define

$$H^+_{2k,\mathrm{alg}} \;\subseteq\; H^+_{2k}$$

to be the subset of classes z which can be represented by a positive holomorphic k-cycle.

Theorem 4.1. *Suppose Conjecture B holds. Then the cycle Γ bounds a positive holomorphic p-chain of mass $\leq \Lambda$ in X if and only if any of the following conditions holds:*

(a) $\widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) \geq -\Lambda$ *for all positive algebraic cycles $Z \subset X - \Gamma$ of codimension-p with cohomology class $\ell[\omega]^p$ for $\ell \in \mathbf{Z}^+$.*

(b) $\widetilde{\mathrm{Wind}}_\lambda(\Gamma, \alpha) \geq -\Lambda$ *for all sparks α satisfying equation (4.1) with Z as above.*

(c) $\frac{1}{p!} \int_\Gamma \beta \geq -\Lambda$ *for all smooth forms $\beta \in \mathcal{E}^{2p-1}(X)$ for which $d^{p,p}\beta + \omega^p \geq 0$ is weakly positive on X.*

(d) *There exists $\tau \in H_{2p}(X, |\Gamma|; \mathbf{Z})$ with $\partial\tau = [\Gamma]$ such that $\tau \bullet [Z] \ \geq\ 0$ for all $[Z]$ as above and $\Lambda = (\tau, \frac{1}{p!}[\omega^p])$*

Proof. Condition (a) represents Theorem 3.2 above. Conditions (a) and (b) are obviously equivalent by equation (4.2). To check the necessity of Condition (c) suppose that $\Gamma = dT$ where T is a positive holomorphic p-chain. Then one has $\int_\Gamma \beta = \int_T d\beta = \int_T d^{p,p}\beta = \int_T (d^{p,p}\beta + \omega^p - \omega^p) \geq -\int_T \omega^p = -p!\mathbf{M}(T)$. On the other hand, Condition (c) implies Condition (a) since the subvarieties Z in question arise by intersection with subvarieties $\widetilde{Z}$ in the ambient projective space, where we can mollify to obtain smooth $(p-1,p-1)$-forms α_ϵ with $d\alpha_\epsilon = \omega^p - \widetilde{Z}_\epsilon$ and $\widetilde{Z}_\epsilon \to \widetilde{Z}$.

For Condition (d), suppose there exists T as above. Then the class $\tau = [T] \in H_{2p}(X, |\Gamma|; \mathbf{Z})$ has the stated properties. Conversely, given τ, choose any $2p$-chain $N \in \tau$. Then $dN = \Gamma$ and for any Z as above we have $0 \le \tau \bullet [Z] = N \bullet Z = N \bullet Z - \ell \int_N \omega^p + \ell \int_N \omega^p = \ell p! \widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) + \ell\tau(\omega^p)$. ∎

Corollary 4.2. *If $H^+_{2(n-p),\mathrm{alg}}$ is contained in a proper subcone of $H^+_{2(n-p)}$, that is, if there exists $\tau \in H_{2p}(X, |\Gamma|; \mathbf{Z})$ with $d\tau \neq 0$ and $\tau \bullet u \ge 0$ for $u \in H^+_{2(n-p),\mathrm{alg}}$, then there exists a positive holomorphic p-chain T on X with non-empty boundary supported in $|\Gamma|$.*

This question of holomorphic representability is discussed in detail in [HL$_5$].

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